Ehrhart polynomials, $h^*$-vectors, and triangulations of matroid polytopes

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Matroid

**Definition**

A non-empty collection $\mathcal{B}$ of subsets of $[n] := \{1, \ldots, n\}$ is the set of bases of a matroid $M$ if and only if

- If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, \exists y \in B_2 \setminus B_1$ such that $(B_1 \cup y) \setminus x \in \mathcal{B}$. 

$B_1$: A base of the graph $G$.  
$B_2$: A base of the graph $G$.  
$B_1 \setminus \{3\} \cup \{1\}$: A base of the graph $G.$
Matroid Examples

**Uniform Matroid**

The **uniform** matroid of rank $r$ on $n$ elements, denoted $U^{r,n}$, has as bases all $r$-subsets of $[n] := \{1, \ldots, n\}$.

**Graphical Matroids**

For a graph $G$ the bases are all spanning trees of $G$.
Matroid Polytope

**Definition**

Let $M$ be a matroid, and $\mathcal{B}(M)$ be its set of bases.

- The **incidence** vector of $B = \{b_1, \ldots, b_r\} \in \mathcal{B}(M)$ is defined as
  \[
  e_B := e_{b_1} + e_{b_2} + \cdots + e_{b_r} \in \mathbb{R}^n
  \]
  where of $e_i$ are the standard unit vectors in $\mathbb{R}^n$.

- The **matroid polytope** $\mathcal{P}(M)$ of $M$ is the convex hull of these points
  \[
  \mathcal{P}(M) := \text{conv}\{ e_B \mid B \in \mathcal{B}(M) \}.
  \]

**Example**

Ex: $e_{\{1,3,4\}} = e_1 + e_3 + e_4 = (1, 0, 1, 1, 0)^\top$. 
Ehrhart Polynomials and $h^*$-vectors

### Ehrhart Polynomial

**Definition**

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be an integral polytope.

- The **Ehrhart polynomial** of $\mathcal{P}$ is the function $i(\mathcal{P}, k) := \# |k \cdot \mathcal{P} \cap \mathbb{Z}^n|$, where $k \in \mathbb{Z}_{\geq 0}$.
- The **Ehrhart series** of $\mathcal{P}$ is $\sum_{k=0}^{\infty} i(\mathcal{P}, k)t^k$.

**Example**

Let $\mathcal{P}$ be the unit square in $\mathbb{R}^2$. Then $i(\mathcal{P}, k) = (k + 1)^2$ and

$$\sum_{k=1}^{\infty} i(\mathcal{P}, k)t^k = \sum_{k=1}^{\infty} (k + 1)^2 t^k.$$
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Introduction

Ehrhart Polynomials and \( h^* \)-vectors

**Theorem**

Let \( P \) be an integral polytope where \( \text{dim}(P) = d \). Then

\[
\sum_{k=0}^{\infty} i(P, k)t^k = \frac{h_d^* t^d + h_{d-1}^* t^{d-1} + \cdots + h_0^*}{(1 - t)^{d+1}}
\]

and \((h_0^*, \ldots, h_d^*)\) is the \( h^* \)-vector of \( P \).

**Example**

For \( P \) the unit square in \( \mathbb{R}^2 \), \( i(P, k) = (k + 1)^2 \) and

\[
\sum_{k=1}^{\infty} i(P, k)t^k = \sum_{k=1}^{\infty}(k + 1)^2 t^k = \frac{1 + t}{(1 - t)^3}. \quad \text{\( h^* \)-vector = (1, 1) }
\]
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Conjectures

$h^*$-vectors and Ehrhart Polynomials

Our Two Conjectures

**Definition**

A vector $(v_1, \ldots, v_n)$ is **unimodal** if $v_1 \leq v_2 \leq \cdots \leq v_i$ and $v_i \geq v_{i+1} \geq v_{i+2} \geq \cdots v_n$ for some $1 \leq i \leq n$.

**Conjecture**

Let $M$ be a matroid on $n$ elements.

(A) The $h^*$-vector of $P(M)$ is unimodal.

(B) The coefficients of the Ehrhart polynomial of $P(M)$ have positive coefficients.
The coefficients of the Ehrhart polynomial of $P(U^2,n)$ are positive.

The $h^*$-vector of $P(U^2,n)$ is unimodal.

Consider $P(U^3,n)$, and let $K$ be a non-negative integer. There exists $n(K) \in \mathbb{N}$ such that for all $n \geq n(K)$ the $h^*$-vector of $P(U^3,n)$, is non-decreasing from index 0 to $K$. That is, $h^*_0 \leq h^*_1 \leq \cdots \leq h^*_K$. 

[Explanation]
Computational Evidence

Strong Evidence

- Verified both conjectures for all uniform matroids up to 75 elements,
- Verified our $h^*$-vector conjecture on over 2500 random realizable matroids over various finite fields,
- Verified our conjectures on many other matroids including the famous 28 matroids presented in Oxley’s book.

See the Matroids!

Generating Functions

**Generating Function**

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope.

$$f(\mathcal{P}) = \sum_{\alpha \in \mathcal{P} \cap \mathbb{Z}^n} z^\alpha,$$

where $z^\alpha = z_{1}^{\alpha_{1}}z_{2}^{\alpha_{2}}\cdots z_{n}^{\alpha_{n}}$.

**Example**

$$f(\mathcal{P}) =
\begin{align*}
x^1y^1 + x^2y^1 + x^3y^1 + x^4y^1 + \\
x^1y^2 + x^2y^2 + x^3y^2 + x^4y^2 + \\
x^1y^3 + x^2y^3 + x^3y^3 + x^4y^3 + x^3y^4
\end{align*}$$

**Note**

$$f(1, 1, \ldots, 1) = \#(\text{Lattice points in } \mathcal{P}).$$
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Techniques and New Results

New Complexity Results

Efficient Computation of Ehrhart Polynomials

**Theorem**

Let $M$ be a matroid on $n$ elements with fixed rank $r$. The Ehrhart polynomial of $P(M)$ can be computed in polynomial time.

**Remark**

In fact the previous theorem holds for integral polymatroids and independence matroid polytopes with bounded rank condition.

**LattE**

LattE is software to compute multivariate generating functions and Ehrhart polynomials.

http://www.math.ucdavis.edu/~latte/
http://www.math.uni-magdeburg.de/~mkoeppe/latte/
### New Directions

**h*-vectors and Unimodal Triangulations**

The $h$-vector of a unimodular triangulation is the $h^*$-vector.

### Variant of White’s Conjecture

Every matroid polytope has a regular unimodular triangulation.

### What is known

- Uniform matroid polytopes have a regular unimodular triangulation. [Sturmfels]
- Graphical matroid polytopes are “close” to having a regular unimodular triangulation. [Blasiak ’05]
All simple matroids with less than 9 elements have a unimodular triangulation.

**Lemma (Sufficient Condition)**

- Let \( M \) be a matroid of rank \( r \) on \( n \) elements and \( T \) a full-dimensional simplex with vertices of \( \mathcal{P}(M) \).
- If \( T \) is connected as an induced subgraph \( G(T) \) of the 1-skeleton of \( \mathcal{P}(M) \), then \( \text{Volume}(T) = 1 \).

**Conjecture**

The matroid polytope \( \mathcal{P}(M) \) can be covered by simplices \( \mathcal{T} \) such that for all \( T \in \mathcal{T} \), \( G(T) \) is connected. Hence \( \mathcal{T} \) is a unimodular covering.
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Thank you for your attention!

http://www.math.ucdavis.edu/~haws/

### Unimodular Triangulations and Coverings

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$h^*$-vector</th>
<th>Ehrhart Polynomial</th>
</tr>
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<tbody>
<tr>
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<tr>
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<td>$P_6$</td>
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<tr>
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<td>$AG(3,2)$</td>
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<td>Non-Pappus</td>
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<td>$Q_3(GF(3)^*)$</td>
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<td>1, 433, 3079, 4193, 594, 167, 601, 787, 19</td>
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<td>$R_9$</td>
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<td>1, 147, 1142, 1717, 656, 60, 1</td>
<td>1, 373, 1140, 5, 1440, 120, 144, 840, 1440</td>
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</tbody>
</table>
### $h^*$-vectors of $P(U_3^n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h^*$-vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(\ast)$</td>
</tr>
<tr>
<td>2</td>
<td>$(\ast, \ast, \cdots, \ast)$</td>
</tr>
<tr>
<td>3</td>
<td>$(\ast, \ast, \ast, \cdots, \ast)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n(K) - 1$</td>
<td>$(\ast, \ast, \ast, \ast, \cdots, \ast)$</td>
</tr>
<tr>
<td>$n(K)$</td>
<td>$\underbrace{(\ast, \ast, \ast, \ast, \ast, \cdots, \ast)}_{0-K}$ Non-decreasing from 0 to $K$</td>
</tr>
<tr>
<td>$n(K) + 1$</td>
<td>$\underbrace{(\ast, \ast, \ast, \ast, \ast, \cdots, \ast)}_{0-K}$ Non-decreasing from 0 to $K$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Back to Theoretical Evidence
Example of Barvinok Rational Functions

Let $\mathcal{P} = \{ x \in \mathbb{R} \mid 0 \leq x \leq 10 \} \subseteq \mathbb{R}$.

We need two rational functions, one for each of the cones $C_1 = \{ x \mid x \geq 0 \}$ and $C_0 = \{ x \mid x \leq 10 \}$.

\[(C_0) \quad \frac{1}{1-x} = 1 + x + x^2 + \cdots \]

\[(C_1) \quad \frac{1}{1-x^{-1}} = 1 + x^{-1} + x^{-2} + \cdots \implies \frac{x^{10}}{1-x^{-1}} = x^{10} + x^{9} + x^{8} + \cdots \]

Sum the two functions

\[f(\mathcal{P}) = \frac{1}{1-x} + \frac{x^{10}}{1-x^{-1}} = \frac{1}{1-x} - \frac{x^{11}}{1-x} = \frac{1 - x^{11}}{1-x} = 1 + x + x^2 + \cdots + x^{10}.\]
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New Directions

Unimodular Triangulations and Coverings

Definition

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be an integral polytope and $\mathbb{F}$ a field.

$A_q(\mathcal{P}) := \mathbb{F} \cdot \{ Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} T^q \mid (\alpha_1, \cdots, \alpha_n) \in q\mathcal{P} \}.$

$A(\mathcal{P}) := \bigoplus_{q=1}^n A_q(\mathcal{P})$ is the Ehrhart Ring of $\mathcal{P}.$

FACT

If $\mathcal{P}$ is an integral polytope and the Ehrhart ring $A(\mathcal{P})$ is Gorrenstein, then the $h^*$-vector of $\mathcal{P}$ is unimodal.

Unimodal yet not Gorrenstein

The matroid polytopes are a case where the Ehrhart ring is not Gorrenstein yet its $h^*$-vector is unimodal for all experiments thus far.